Braess Paradox in Electrical Networks – When more might mean less

J. Rojas, A. Alonso and D. Quesada

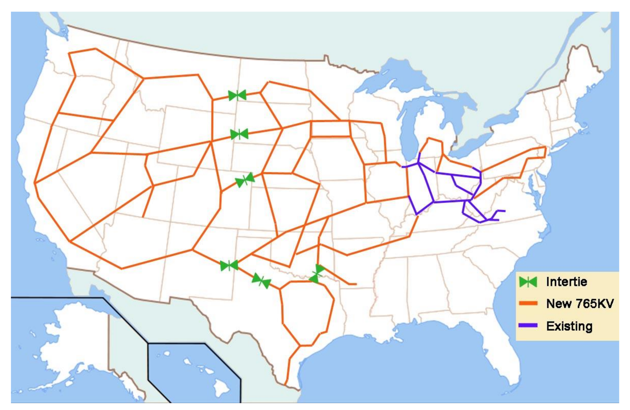
School of Science, Technology and Engineering Management

St. Thomas University, Miami Gardens, FL 33054

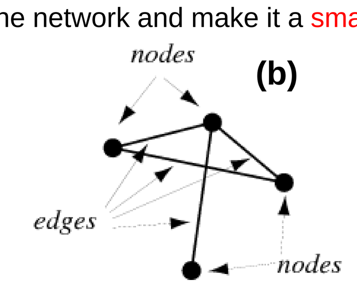
Abstract

Electrical grids are part of the network of connections maintaining a city alive nowadays. Many times we see a limited amount of lines and poles, as well as, supportive local electrical transformer points (LETP). Most of these connections are wired in parallel in order to guarantee a sustainable flow of electricity plus being robust enough against failures. Why are we not making the system redundant and increasing the number of grid points and cabling? Despite of the economic cost of such approach there is a counter intuitive fact known as the Braess Paradox, stating that the addition of some extra lines will be detrimental to the performance level of the network or the grid. This fact is extremely important when you are designing Smart Grids and Cities. In this presentation, as part of a term project for MAD 3300 Graph Theory and Networks the Braess Paradox is investigated for several network configurations. Special interest is dedicated to the Wheatstone bridge and to those networks containing such configurations as part of their structural elements. The flow across the network as well as the overall resistance are computed and expressed in terms of network characteristics.

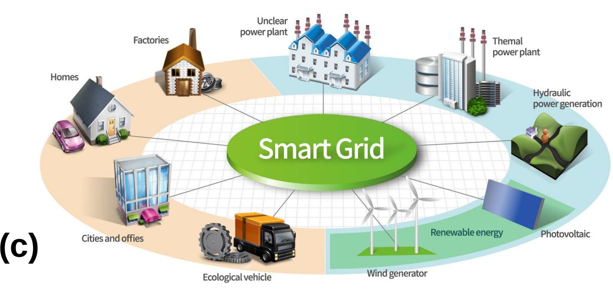
1. Motivations and General Ideas



(a)



(b)



(c)

Fig 1: Schematic representation of the USA backbone electrical grid (a). As it might be notice, the grid can be mapped into a graph or network (b), where the hubs are vertices and power lines are edges. The idea is to optimize the topology of the network and make it a smart grid (c).

What is Braess Paradox?

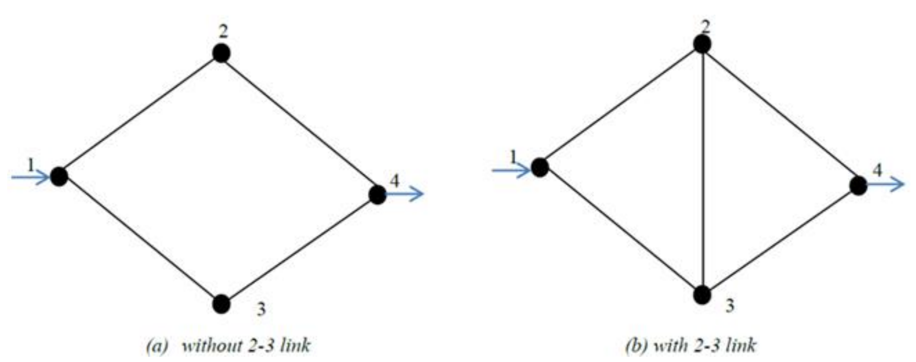


Fig 2: Schematic representation of the Braess Paradox when four vertices are considered and an extra edge is added to the grid.

Braess paradox states that in a congested network, it may happen that adding a new path between destinations can increase the level of congestion. In transportation networks, the phenomenon results from the decisions of network participants who selfishly seek to optimize their own performance metrics. In an electric power distribution network, an analogous increase in congestion can arise as a consequence of **Kirchhoff's laws**.

What is Wheatstone Bridge?

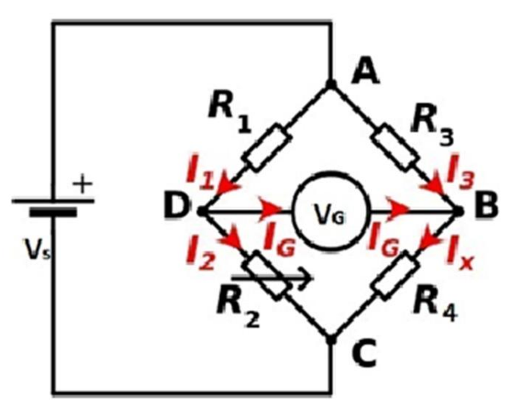


Fig 3: The Wheatstone bridge is an electrical bridge circuit used to measure resistance.

- This bridge consists of a galvanometer and two parallel branches containing four resistors. One parallel branch contains one fixed resistance R3 and one unknown R4; the other parallel branch contains an adjustable resistance R2 and finally another fixed resistance R1.
- Due to this specific arrangement of resistors, we observe that the resistance of both arms of the bridge circuit is the same. Thus, by applying Kirchhoff's laws, we obtain the equation which shows the relationship of the resistance between the two arms of the bridge:

$$R_4 = R_3 \frac{R_2}{R_1}$$

2. Objectives

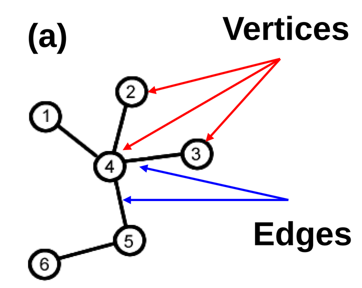
- Learn how to map electrical grids into mathematical networks and characterize them for structural and functional optimization.
- Learn about Braess Paradox and networks with Wheatstone Bridge configurations.
- Observe the presence of Braess Paradox in Wheatstone networks due to unmeet conditions.
- Developing an algorithm for detecting embedded Wheatstone subnetworks.
- Observe how extra Wheatstone connections in the network makes the system unbalanced, thus causing congestion in the flow across the network

3. Model Implementation

Foundations of Graph Theory

Fig 4 : (a) In Graph Theory, a Graph is defined as the collection of vertices or points that are connected by edges or lines, (b) By looking at the configuration of a graph, we can determine the number of edges connected to its vertices or known formally as the degree of its vertices.

(a)



(b)

Vertex	Degree
1	1
2	1
3	1
4	4
5	2
6	1

Fig 5 : The idea of connections can be extended to the concept of adjacency, where two vertices are joined by an edge, and consequently, we say that these two vertices are incident with that edge that bonds both together or vice versa.

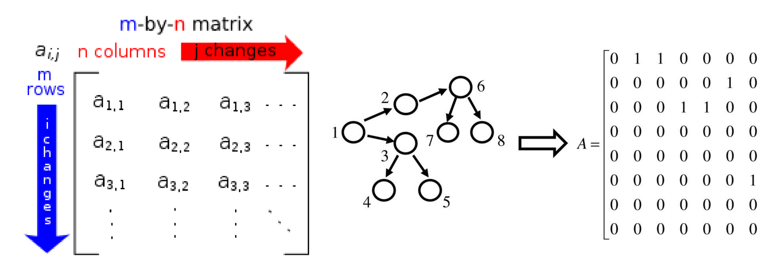
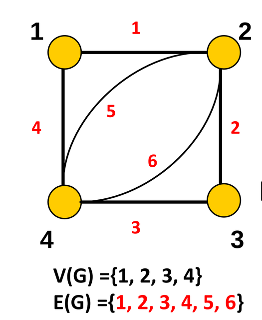


Fig 6 : (a) Particularly, we can represent any graph based on the previous information with either its corresponding adjacency or incidence matrix. If n represents the number of vertices and m represents the number of edges, the adjacency matrix will have dimensions of **nxn** in which the entry in row **i** and column **j** is the number of edges joining the vertices **i** and **j**. On the other hand, the incidence matrix is the **nxm** matrix in which the entry in row **i** and column **j** is 1 if vertex **i** is incident with edge **j**, and 0 otherwise.

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

Adjacency Matrix



V(G) ={1, 2, 3, 4}
E(G) ={1, 2, 3, 4, 5, 6}

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Incidence Matrix

Applied Graph Theory / Wheatstone Detection Algorithm

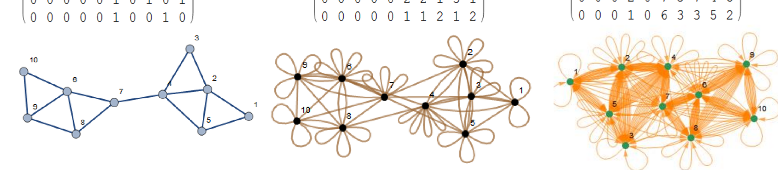
Mathematica is a symbolic mathematical computation al program, sometimes called a computer algebra program, used in many scientific, engineering, mathematical, and computing fields.

Step 1. Calculate the node-node adjacency matrices N, N^2, and N^3 for the reduced-form 13-bus network.

$$N = \begin{pmatrix} 0&1&0&0&1&0&0&0&0&0 \\ 1&0&1&1&1&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0&0 \\ 0&1&1&0&1&0&1&0&0&0 \\ 1&1&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1&1 \\ 0&0&0&1&0&1&0&1&0&0 \\ 0&0&0&0&0&1&1&0&1&0 \\ 0&0&0&0&0&1&0&1&0&1 \\ 0&0&0&0&0&1&0&0&1&0 \end{pmatrix}$$

$$N^2 = \begin{pmatrix} 2&1&1&2&1&0&0&0&0&0 \\ 1&4&1&2&2&0&1&0&0&0 \\ 1&1&2&1&2&0&1&0&0&0 \\ 2&2&1&4&1&1&0&1&0&0 \\ 1&2&2&1&3&0&1&0&0&0 \\ 0&0&0&1&0&4&1&2&2&1 \\ 0&1&1&0&1&1&3&1&2&1 \\ 0&0&0&1&0&2&1&3&1&2 \\ 0&0&0&0&0&2&2&1&3&1 \\ 0&0&0&0&0&1&1&2&1&2 \end{pmatrix}$$

$$N^3 = \begin{pmatrix} 2&6&3&3&5&0&2&0&0&0 \\ 6&6&6&8&7&1&2&1&0&0 \\ 3&6&2&6&3&1&1&1&0&0 \\ 3&8&6&4&8&1&6&1&2&1 \\ 5&7&3&8&4&1&1&1&0&0 \\ 0&1&1&1&1&6&7&7&7&6 \\ 2&2&1&6&1&7&2&6&3&3 \\ 0&1&1&1&1&7&6&4&7&3 \\ 0&0&0&2&0&7&3&7&4&5 \\ 0&0&0&1&0&6&3&3&5&2 \end{pmatrix}$$



Step 2. Construct the diagonal entries of the T Matrix. To obtain the triangles, you must divide the diagonal entries by $\frac{1}{2}$.

(a)

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&3&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&2&0&0&0&0&0&0 \\ 0&0&0&0&2&0&0&0&0&0 \\ 0&0&0&0&0&3&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&2&0&0 \\ 0&0&0&0&0&0&0&0&2&0 \\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

(b)

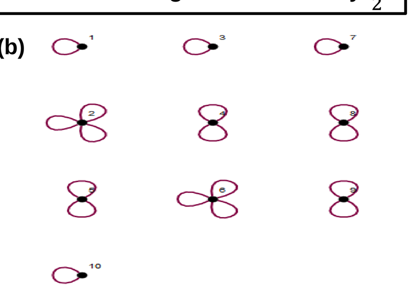


Fig 7: (a) Shows the construction of matrix T. The diagonal of the matrix demonstrates the number of triangles connected to each bus. (b) Graphical representation of matrix T. Buses 1-10 have loops, which we will use to represent the number of triangles connected to each bus

Step 3. Construct a matrix Ñ1 from Matrix N and a 10x10 matrix of ones ,Matrix 1. Ñ1 = 1- N

(a.)

$$\begin{pmatrix} 1&0&1&1&0&1&1&1&1&1 \\ 0&1&0&0&0&1&1&1&1&1 \\ 1&0&1&0&1&1&1&1&1&1 \\ 1&0&0&1&0&1&0&1&1&1 \\ 0&0&1&0&1&1&1&1&1&1 \\ 1&1&1&1&1&1&0&0&0&0 \\ 1&1&1&0&1&0&1&0&1&1 \\ 1&1&1&1&1&0&0&1&0&1 \\ 1&1&1&1&1&0&1&0&1&0 \\ 1&1&1&1&1&0&1&1&0&1 \end{pmatrix}$$

(b.)

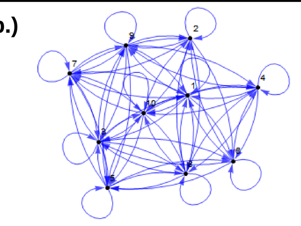


Fig 8 : (a.) Matrix Ñ1's entries will tell us which pairs of nodes have geodesic paths of length equal to one. This matrix will be used for solving the indicator matrix. (b.) Graph showing the geodesic paths of length 1.

Step 4. The Pairs of nodes in the network with a geodesic path of length one correspond to the off - diagonal nonzero entries of NG2

(a.)

$$\begin{pmatrix} 2&0&1&2&0&0&0&0&0&0 \\ 0&4&0&0&0&0&1&0&0&0 \\ 0&0&2&0&2&0&1&0&0&0 \\ 0&0&0&4&0&1&0&1&0&0 \\ 0&0&0&0&3&0&1&0&0&0 \\ 0&0&0&0&0&4&0&0&0&0 \\ 0&0&0&0&0&0&3&0&2&1 \\ 0&0&0&0&0&0&0&3&0&2 \\ 0&0&0&0&0&0&0&0&3&0 \\ 0&0&0&0&0&0&0&0&0&2 \end{pmatrix}$$

(b.)

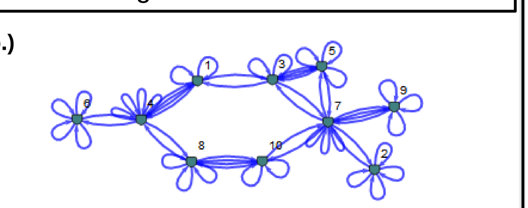
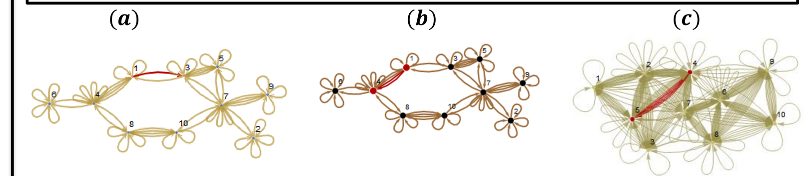


Fig 9: (a) Matrix NG2 demonstrates 1 or 2 geodesic paths of length one within the network. Since the matrix is symmetric, we will only be looking at the upper triangular part of the matrix. (b.) NG2 graph will be used to demonstrate sets R1 and R2.

Step 5. Define R^1, R^2, R^3



(a) {{1, 3}, {1, 4}, {2, 7}, {3, 5}, {3, 7}, {4, 6}, {4, 8}, {5, 7}, {7, 9}, {7, 10}, {8, 10}}

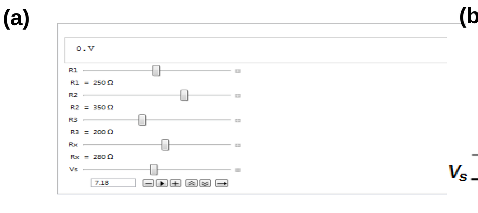
(b) {{1, 4}, {3, 5}, {7, 9}, {8, 10}}

(c) {{1, 2}, {1, 3}, {1, 4}, {1, 5}, {1, 7}, {2, 3}, {2, 4}, {2, 5}, {2,7}, {3, 4}, {3, 5}, {4, 5}, {4, 7}, {4, 9}, {6, 7}, {6, 8}, {6, 9}, {6, 10}, {7, 8}, {7, 9}, {7, 10}, {8, 9}, {8, 10}, {9, 10}}

Fig 10: (a) Defines R1 as the set of all pairs of nodes that have one or two geodesic paths of length equal to one. (b) Defines R2 as the set of all pairs of nodes that have exactly two geodesic paths of length equal to one. Defines R3 as the set of all pairs of nodes that are separated by at least two paths of length 1.

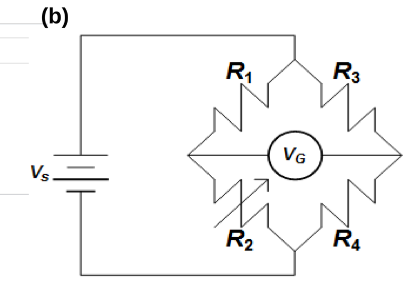
Fig 11 : (a) is a simulation which was created using Mathematica too balance the bridge out. We created a function too see whether it is balanced or not. (b) is a graphical demonstrating a Wheatstone bridge.

(a)




```
Balanced[R1_, R2_, R3_, Rx_] := R2 / R1 == Rx / R3
Balanced[250, 350, 200, 280]
True
```

(b)



4. Numerical & Graphical Solutions

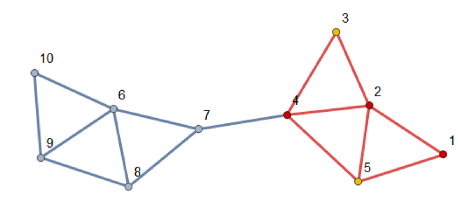
- Define *WS* = *T* ∩ *D* ∩ *R2* ∩ *R3* . Some care is required in defining the intersection of these sets, since *T* and *D* contain a list of nodes, whereas *R2* and *R3* contain a list of pairs of nodes. If Ω is a set of single elements, and Ψ is a set of pairs of elements, then we will say that $\{\psi_i,\psi_j\} \in \Omega \cap \Psi$ if and only if $\psi_i \in \Omega$ and $\psi_j \in \Omega$, for all ψ_i, $\psi_j \in \Psi$
- Calculate the clustering coefficient for all the sub-graphs. Those for which K = 5/6 represent Wheatstone sub-networks.

$$K_{wheatstone} = \frac{1}{4}\left(1 + \frac{2}{3} + \frac{2}{3} + 1\right) = \frac{5}{6}$$

- For all pairs of nodes {(*i*,*j*)} in *WS*, construct the node-node adjacency matrix for the sub-graph consisting of *i*, *j*, and all nodes that are neighbors of both *i* and *j* (that is, those nodes which have a geodesic path distance of one from both *i* and *j*). Ignore any direct links between *i* and *j*. We take the union of the T matrix, degrees of the nodes, R2 and R3. By organizing the data into a series of 4 tables, we were able to conclude upon analyzing the reduced network graph that each table gives us each of the Wheatstone configurations present.

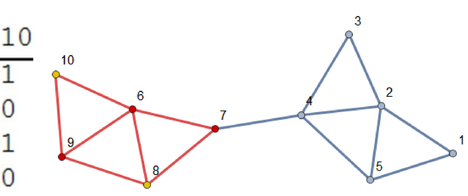
Wheatstone Bridge 1

	1	2	4	5
1	0	1	0	1
2	1	0	1	1
4	0	1	0	1
5	1	1	1	0



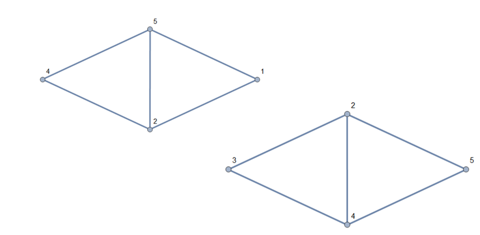
Wheatstone bridge 3

	6	8	9	10
6	0	1	1	1
8	1	0	1	0
9	1	1	0	1
10	1	0	1	0



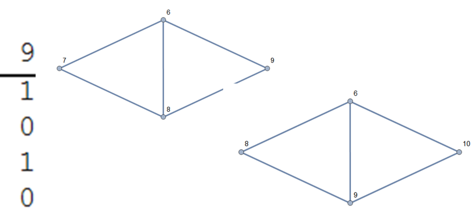
Wheatstone bridge 2

	2	3	4	5
2	0	1	1	1
3	1	0	1	0
4	1	1	0	1
5	1	0	1	0



Wheatstone bridge 4

	6	7	8	9
6	0	1	1	1
7	1	0	1	0
8	1	1	0	1
9	1	0	1	0



Demonstration of Braess Paradox

Fig 12: The reduced network graph where the highlighted lines represent the shortest and convenient path from one node to any other that traverses the least amount of resistances present . These are labeled with "+" while all others are indicated with a "-".

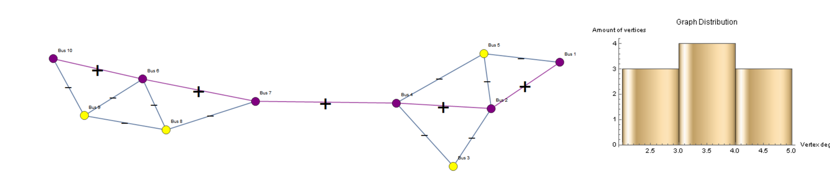


Fig 13: Upon creating the 13 - bus system with the added edges to the reduced network, we can see that based on the distribution of "+" & "-" lines in both network, the reduced network is balanced, but the 13 - bus network is unbalanced. Now, by comparing the values of the total resistance traversed between both networks, we see that the reduced network' s path was more efficient in avoiding the larger resistances.

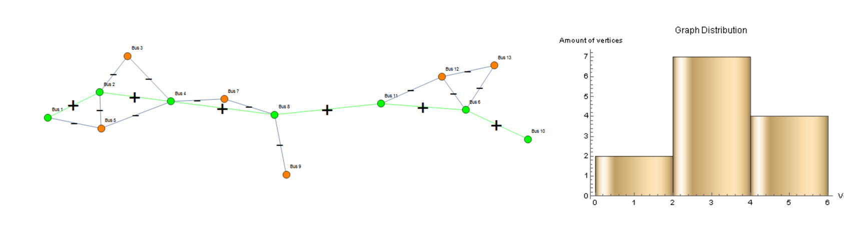
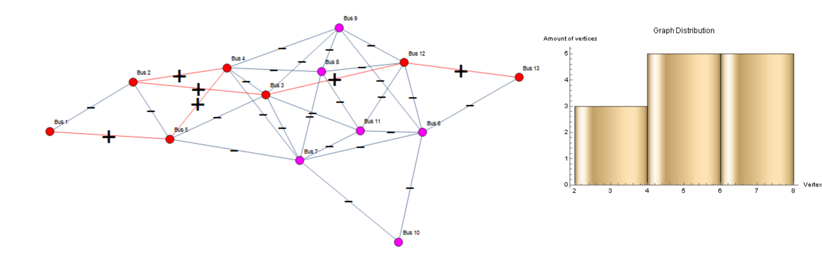


Fig 14: Too demonstrate how adding more edges to the network, would increase the amount of congestion. We created a 13-bus network with 31 connections. Actually, the more edges you add to the network, the more unbalanced the system becomes. Even if you increase the amount of buses, the more complex the network, the more unbalanced it will become.



5. Conclusion

- **Braess Paradox, a concept originally involved with traffic networks and its counterintuitive approach of adding an additional road to alleviate congestion, can be extended to its implications with electrical grids.**
- **Specifically, the consequences of erroneously increasing the number of grid points and cablings for better transmission of electrical power in a network may actually decrease the networks level of performance and lead to detrimental loses in electrical power flow and ultimately cause power outages across the grid.**

6. Acknowledgements

Special thanks to our professor David Quesada for introducing us to the concepts involved in Graph Theory and how these concepts can be applied to various areas of study and world problems. His endless effort, enthusiasm and passion towards this project's topic was crucial in our satisfactory completion of this project. In addition, we greatly appreciate being assigned a project such as this one as it gave us a clear picture of how impactful Graph Theory can be in our daily lives.

7. References

[1] Blumsack, S., 2006. "Network Topologies and Transmission Investment Under Electric-Industry Restructuring," unpublished Ph.D. dissertation, Department of Engineering and Public Policy, Carnegie Mellon University. Available at www.andrew.cmu.edu/~sblumsac.

[2] Blumsack S. and Ilic M., "The Braess Paradox in Electric Power Systems",http://www.personal.psu.edu/sab51/braess ´ paradox.pdf